Dynamic Programming

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Topic Overview

- Overview of Serial Dynamic Programming
- Serial Monadic DP Formulations
- Nonserial Monadic DP Formulations
- Serial Polyadic DP Formulations
- Nonserial Polyadic DP Formulations

Overview of Serial Dynamic Programming

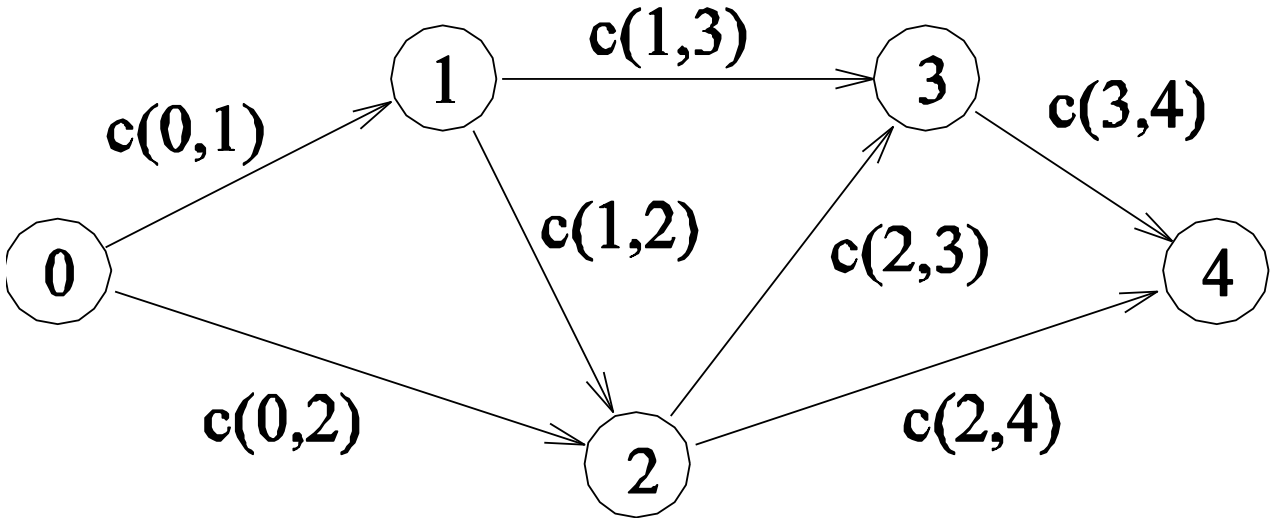
- *Dynamic programming* (DP) is used to solve a wide variety of discrete optimization problems such as scheduling, string-editing, packaging, and inventory management.
- Break problems into subproblems and combine their solutions into solutions to larger problems.
- In contrast to divide-and-conquer, there may be relationships across subproblems.

Dynamic Programming: Example

- Consider the problem of finding a shortest path between a pair of vertices in an acyclic graph.
- An edge connecting node *i* to node *j* has cost *c(i,j)*.
- The graph contains *n* nodes numbered *0,1,…, n-1*, and has an edge from node *i* to node *j* only if *i* < *j*. Node 0 is source and node *n-1* is the destination.
- Let *f(x)* be the cost of the shortest path from node 0 to node *x*.

$$f(x) = \begin{cases} 0 & x = 0 \\ \min\limits_{0 \le j < x} \{f(j) + c(j,x)\} & 1 \le x \le n-1 \end{cases}$$

Dynamic Programming: Example



- A graph for which the shortest path between nodes 0 and 4 is to be computed.

$$f(4) = \min\{f(3) + c(3,4), f(2) + c(2,4)\}.$$

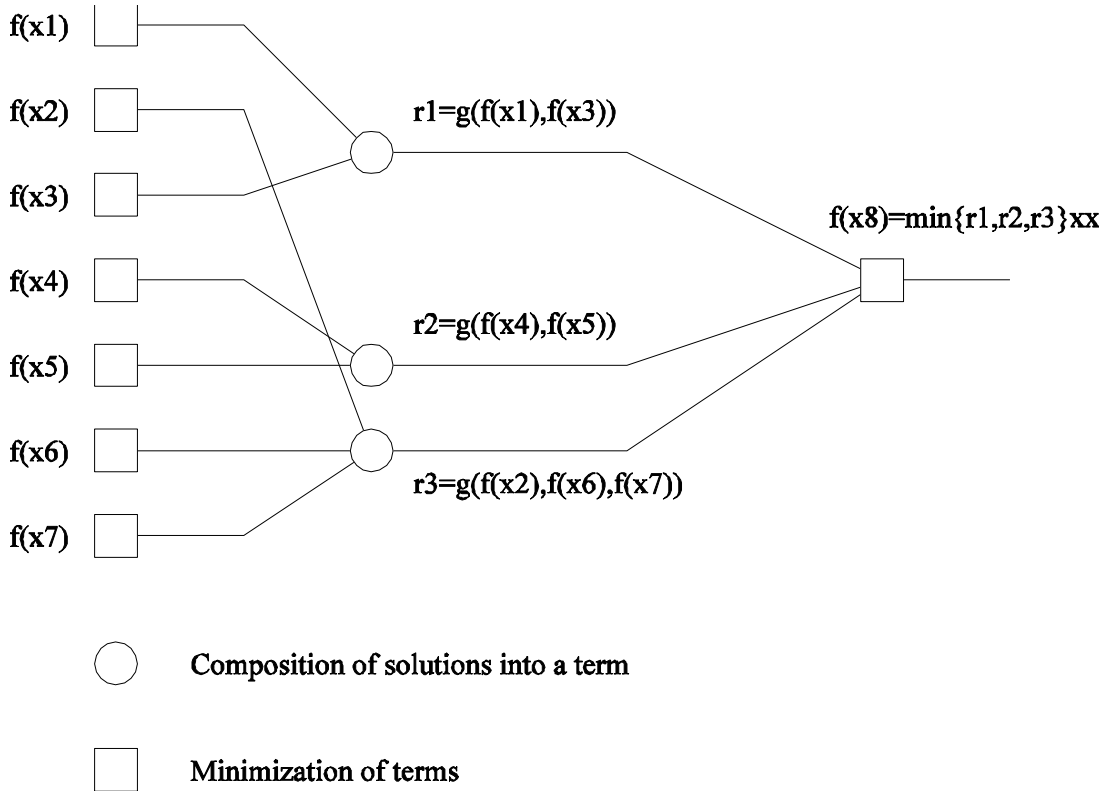
Dynamic Programming

- The solution to a DP problem is typically expressed as a minimum (or maximum) of possible alternate solutions.
- If *r* represents the cost of a solution composed of subproblems $x_1, x_2, \ldots, x_l$, then *r* can be written as

$$r = g(f(x_1), f(x_2), \ldots, f(x_l)).$$

  Here, *g* is the *composition function*.
- If the optimal solution to each problem is determined by composing optimal solutions to the subproblems and selecting the minimum (or maximum), the formulation is said to be a DP formulation.

Dynamic Programming: Example



The computation and composition of subproblem solutions to solve problem $f(x_8)$.

Dynamic Programming

- The recursive DP equation is also called the *functional equation* or *optimization equation*.
- In the equation for the shortest path problem the composition function is *f(j) + c(j,x)*. This contains a single recursive term (*f(j)*). Such a formulation is called monadic.
- If the RHS has multiple recursive terms, the DP formulation is called polyadic.

Dynamic Programming

- The dependencies between subproblems can be expressed as a graph.
- If the graph can be levelized (i.e., solutions to problems at a level depend only on solutions to problems at the previous level), the formulation is called serial, else it is called non-serial.
- Based on these two criteria, we can classify DP formulations into four categories - serial-monadic, serial-polyadic, non-serial-monadic, non-serial-polyadic.
- This classification is useful since it identifies concurrency and dependencies that guide parallel formulations.

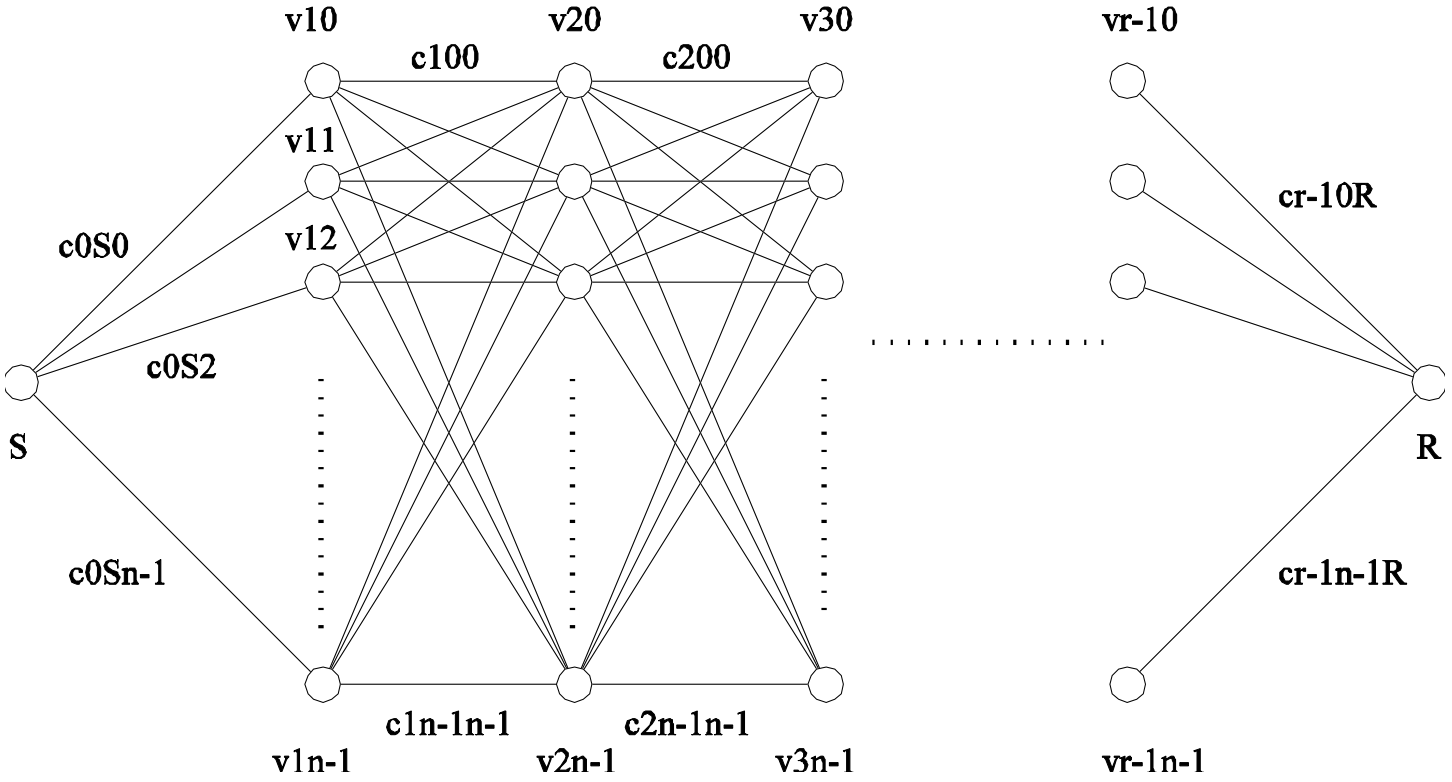
Serial Monadic DP Formulations

- It is difficult to derive canonical parallel formulations for the entire class of formulations.
- For this reason, we select two representative examples, the shortest-path problem for a multistage graph and the 0/1 knapsack problem.
- We derive parallel formulations for these problems and identify common principles guiding design within the class.

Shortest-Path Problem

- Special class of shortest path problem where the graph is a weighted multistage graph of $r + 1$ levels.
- Each level is assumed to have n levels and every node at level i is connected to every node at level $i + 1$.
- Levels zero and r contain only one node, the source and destination nodes, respectively.
- The objective of this problem is to find the shortest path from S to R.

Shortest-Path Problem



An example of a serial monadic DP formulation for finding the shortest path in a graph whose nodes can be organized into levels.

Shortest-Path Problem

- The i^{th} node at level l in the graph is labeled v_i^l and the cost of an edge connecting v_i^l to node v_j^{l+1} is labeled $c_{i,j}^l$.
- The cost of reaching the goal node R from any node v_i^l is represented by C_i^l.
- If there are n nodes at level l, the vector
  $[C_0^l, C_1^{l,\ldots,} C_{n-1}^l]^T$ is referred to as C^l. Note that
  $C_0 = [C_0^0]$.
- We have $C_i^l = min\ \{(c_{i,j}^l + C_j^{l+1}) \mid j$ is a node at level $l + 1\}$

Shortest-Path Problem

- Since all nodes v_j^{r-1} have only one edge connecting them to the goal node R at level r, the cost C_j^{r-1} is equal to $c_{j,R}^{r,1}$.
- We have:

$$C^{r-1} = [c_{0,R}^{r-1}, c_{1,R}^{r-1}, \ldots, c_{n-1,R}^{r-1}].$$

Notice that this problem is serial and monadic.

Shortest-Path Problem

- The cost of reaching the goal node *R* from any node at level *l* is *(0 < l < r – 1)* is

$$C_0^l = \min\{(c_{0,0}^l + C_0^{l+1}), (c_{0,1}^l + C_1^{l+1}), \ldots, (c_{0,n-1}^l + C_{n-1}^{l+1})\},$$

$$C_1^l = \min\{(c_{1,0}^l + C_0^{l+1}), (c_{1,1}^l + C_1^{l+1}), \ldots, (c_{1,n-1}^l + C_{n-1}^{l+1})\},$$

$$\vdots$$

$$C_{n-1}^l = \min\{(c_{n-1,0}^l + C_0^{l+1}), (c_{n-1,1}^l + C_1^{l+1}), \ldots, (c_{n-1,n-1}^l + C_{n-1}^{l+1})\}.$$

Shortest-Path Problem

- We can express the solution to the problem as a modified sequence of matrix-vector products.
- Replacing the addition operation by minimization and the multiplication operation by addition, the preceding set of equations becomes:

$$C^l = M_{l,l+1} \times C^{l+1},$$

where C^l and C^{l+1} are $n \times 1$ vectors representing the cost of reaching the goal node from each node at levels l and $l + 1$.

Shortest-Path Problem

- Matrix $M_{l,l+1}$ is an *n x n* matrix in which entry *(i, j)* stores the cost of the edge connecting node *i* at level *l* to node *j* at level *l* + *1*.

$$M_{l,l+1} = \begin{bmatrix} c^l_{0,0} & c^l_{0,1} & \cdots & c^l_{0,n-1} \\ c^l_{1,0} & c^l_{1,1} & \cdots & c^l_{1,n-1} \\ \vdots & \vdots & & \vdots \\ c^l_{n-1,0} & c^l_{n-1,1} & \cdots & c^l_{n-1,n-1} \end{bmatrix}.$$

- The shortest path problem has been formulated as a sequence of *r* matrix-vector products.

Parallel Shortest-Path

- We can parallelize this algorithm using the parallel algorithms for the matrix-vector product.
- $\Theta(n)$ processing elements can compute each vector C^l in time $\Theta(n)$ and solve the entire problem in time $\Theta(rn)$.
- In many instances of this problem, the matrix M may be sparse. For such problems, it is highly desirable to use sparse matrix techniques.

0/1 Knapsack Problem

- We are given a knapsack of capacity *c* and a set of *n* objects numbered *1,2,…,n*. Each object *i* has weight w_i and profit p_i.
- Let $v = [v_1, v_2, \ldots, v_n]$ be a solution vector in which $v_i = 0$ if object *i* is not in the knapsack, and $v_i = 1$ if it is in the knapsack.
- The goal is to find a subset of objects to put into the knapsack so that

$$\sum_{i=1}^{n} w_i v_i \leq c$$

(that is, the objects fit into the knapsack) and

$$\sum_{i=1}^{n} p_i v_i$$

is maximized (that is, the profit is maximized).

0/1 Knapsack Problem

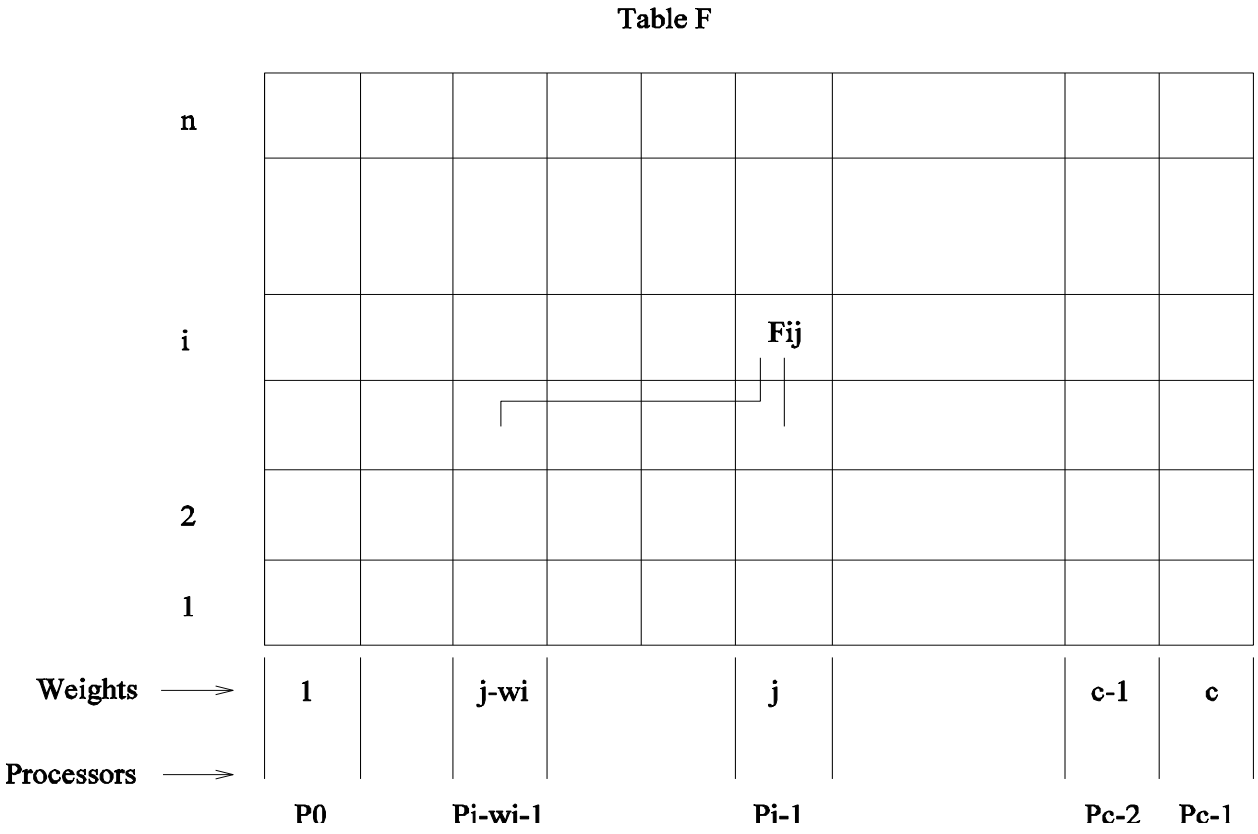
- The naive method is to consider all 2^n possible subsets of the n objects and choose the one that fits into the knapsack and maximizes the profit.
- Let $F[i,x]$ be the maximum profit for a knapsack of capacity x using only objects $\{1,2,\ldots,i\}$. The DP formulation is:

$$F[i,x] = \begin{cases} 0 & x \geq 0, i = 0 \\ -\infty & x < 0, i = 0 \\ \max\{F[i-1,x], (F[i-1,x-w_i] + p_i)\} & 1 \leq i \leq n \end{cases}$$

0/1 Knapsack Problem

- Construct a table *F* of size *n x c* in row-major order.
- Filling an entry in a row requires two entries from the previous row: one from the same column and one from the column offset by the weight of the object corresponding to the row.
- Computing each entry takes constant time; the sequential run time of this algorithm is $\Theta(nc)$.
- The formulation is serial-monadic.

0/1 Knapsack Problem



Computing entries of table *F* for the 0/1 knapsack problem. The computation of entry *F[i,j]* requires communication with processing elements containing entries *F[i-1,j]* and *F[i-1,j-w_i]*.

0/1 Knapsack Problem

- Using *c* processors in a PRAM, we can derive a simple parallel algorithm that runs in *O(n)* time by partitioning the columns across processors.
- In a distributed memory machine, in the j^{th} iteration, for computing *F[j,r]* at processing element P_{r-1}, *F[j-1,r]* is available locally but $F[j-1,r-w_j]$ must be fetched.
- The communication operation is a circular shift and the time is given by $(t_s + t_w) \log c$. The total time is therefore $t_c + (t_s + t_w) \log c$.
- Across all *n* iterations (rows), the parallel time is $O(n \log c)$. Note that this is not cost optimal.

0/1 Knapsack Problem

- Using p-processing elements, each processing element computes c/p elements of the table in each iteration.
- The corresponding shift operation takes time $(2t_s + t_w c/p)$, since the data block may be partitioned across two processors, but the total volume of data is c/p.
- The corresponding parallel time is $n(t_c c/p + 2t_s + t_w c/p)$, or $O(nc/p)$ (which is cost-optimal).
- Note that there is an upper bound on the efficiency of this formulation.

Nonserial Monadic DP Formulations: Longest-Common-Subsequence

- Given a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$, a subsequence of A can be formed by deleting some entries from A.
- Given two sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, find the longest sequence that is a subsequence of both A and B.
- If $A = \langle c,a,d,b,r,z \rangle$ and $B = \langle a,s,b,z \rangle$, the longest common subsequence of A and B is $\langle a,b,z \rangle$.

Longest-Common-Subsequence Problem

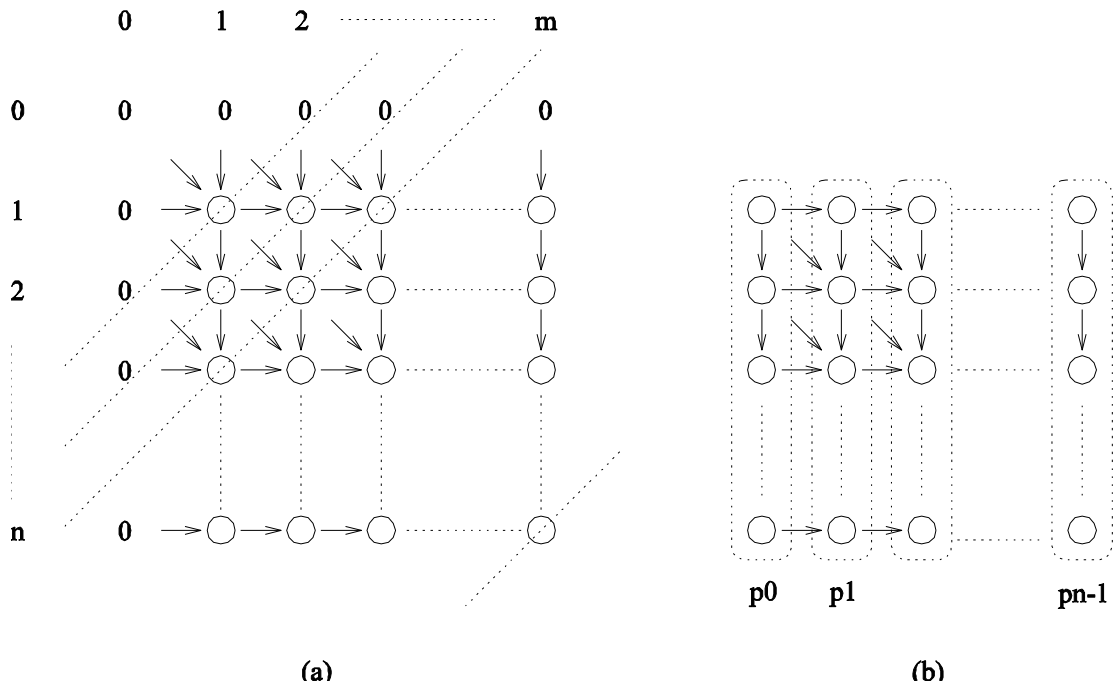
- Let *F[i,j]* denote the length of the longest common subsequence of the first *i* elements of *A* and the first *j* elements of *B*. The objective of the LCS problem is to find *F[n,m]*.
- We can write:

$$F[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ F[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{F[i, j-1], F[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

Longest-Common-Subsequence Problem

- The algorithm computes the two-dimensional *F* table in a row- or column-major fashion. The complexity is $\Theta(nm)$.
- Treating nodes along a diagonal as belonging to one level, each node depends on two subproblems at the preceding level and one subproblem two levels prior.
- This DP formulation is nonserial monadic.

Longest-Common-Subsequence Problem



(a) Computing entries of table for the longest-common-subsequence problem. Computation proceeds along the dotted diagonal lines. (b) Mapping elements of the table to processing elements.

Longest-Common-Subsequence: Example

- Consider the LCS of two amino-acid sequences H E A G A W G H E E and P A W H E A E. For the interested reader, the names of the corresponding amino-acids are A: Alanine, E: Glutamic acid, G: Glycine, H: Histidine, P: Proline, and W: Tryptophan.

		H	E	A	G	A	W	G	H	E	E
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	0	0	0	0	0	0	0	0	0
A	0	0	0	1	1	1	1	1	1	1	1
W	0	0	0	1	1	1	2	2	2	2	2
H	0	1	1	1	1	1	2	2	3	3	3
E	0	1	2	2	2	2	2	2	3	4	4
A	0	1	2	3	3	3	3	3	3	4	4
E	0	1	2	3	3	3	3	3	3	4	5

- The *F* table for computing the LCS of the sequences. The LCS is A W H E E.

Parallel Longest-Common-Subsequence

- Table entries are computed in a diagonal sweep from the top-left to the bottom-right corner.
- Using *n* processors in a PRAM, each entry in a diagonal can be computed in constant time.
- For two sequences of length *n*, there are *2n-1* diagonals.
- The parallel run time is $\Theta(n)$ and the algorithm is cost-optimal.

Parallel Longest-Common-Subsequence

- Consider a (logical) linear array of processors. Processing element P_i is responsible for the $(i+1)^{th}$ column of the table.
- To compute *F[i,j]*, processing element P_{j-1} may need either *F[i-1,j-1]* or *F[i,j-1]* from the processing element to its left. This communication takes time $t_s + t_w$.
- The computation takes constant time (t_c).
- We have:

$$T_P = (2n - 1)(t_s + t_w + t_c).$$

- Note that this formulation is cost-optimal, however, its efficiency is upper-bounded by 0.5!
- Can you think of how to fix this?

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- Given weighted graph $G(V,E)$, Floyd's algorithm determines the cost $d_{i,j}$ of the shortest path between each pair of nodes in V.
- Let $d_{i,j}^k$ be the minimum cost of a path from node i to node j, using only nodes $v_0, v_1, \ldots, v_{k-1}$.
- We have:

$$d_{i,j}^k = \begin{cases} c_{i,j} & k = 0 \\ \min\{d_{i,j}^{k-1}, (d_{i,k}^{k-1} + d_{k,j}^{k-1})\} & 0 \leq k \leq n-1 \end{cases}.$$

- Each iteration requires time $\Theta(n^2)$ and the overall run time of the sequential algorithm is $\Theta(n^3)$.

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- A PRAM formulation of this algorithm uses n^2 processors in a logical 2D mesh. Processor $P_{i,j}$ computes the value of $d_{i,j}^k$ for $k=1,2,\ldots,n$ in constant time.
- The parallel runtime is $\Theta(n)$ and it is cost-optimal.
- The algorithm can easily be adapted to practical architectures, as discussed in our treatment of Graph Algorithms.

Nonserial Polyadic DP Formulation: Optimal Matrix-Parenthesization Problem

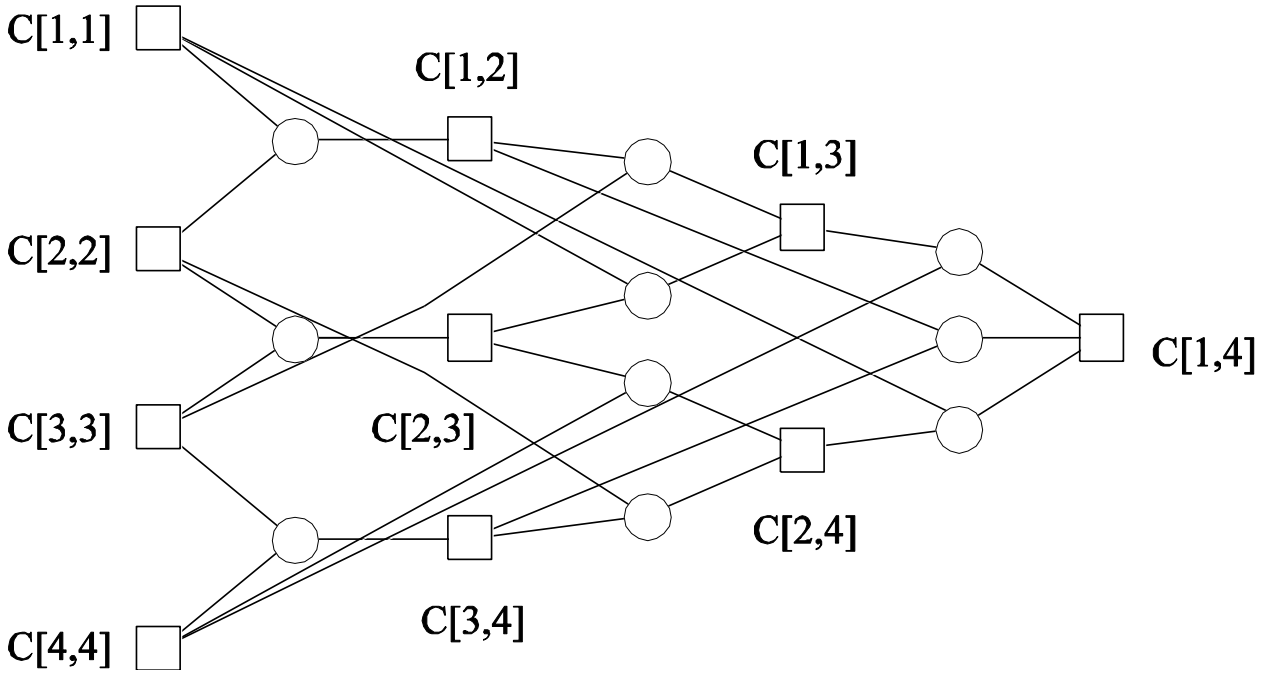
- When multiplying a sequence of matrices, the order of multiplication significantly impacts operation count.
- Let $C[i,j]$ be the optimal cost of multiplying the matrices $A_i, \ldots A_j$.
- The chain of matrices can be expressed as a product of two smaller chains, $A_i, A_{i+1}, \ldots, A_k$ and $A_{k+1}, \ldots, A_j$.
- The chain $A_i, A_{i+1}, \ldots, A_k$ results in a matrix of dimensions $r_{i-1} \times r_k$, and the chain $A_{k+1}, \ldots, A_j$ results in a matrix of dimensions $r_k \times r_j$.
- The cost of multiplying these two matrices is $r_{i-1} r_k r_j$.

Optimal Matrix-Parenthesization Problem

- We have:

$$C[i,j] = \begin{cases} \min\limits_{i \le k < j} \{C[i,k] + C[k+1,j] + r_{i-1} r_k r_j\} & 1 \le i < j \le n \\ 0 & j = i, 0 < i \le n \end{cases}$$

Optimal Matrix-Parenthesization Problem



A nonserial polyadic DP formulation for finding an optimal matrix parenthesization for a chain of four matrices. A square node represents the optimal cost of multiplying a matrix chain. A circle node represents a possible parenthesization.

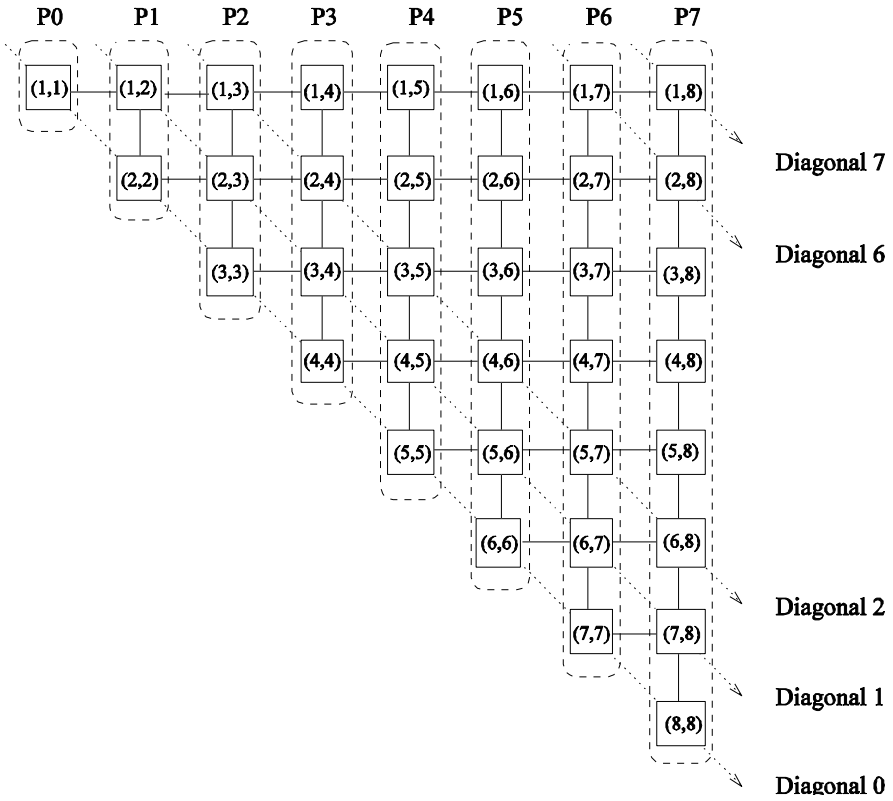
Optimal Matrix-Parenthesization Problem

- The goal of finding *C[1,n]* is accomplished in a bottom-up fashion.
- Visualize this by thinking of filling in the *C* table diagonally. Entries in diagonal *l* corresponds to the cost of multiplying matrix chains of length *l+1*.
- The value of *C[i,j]* is computed as $min\{C[i,k] + C[k+1,j] + r_{i-1}r_k r_j\}$, where *k* can take values from *i* to *j-1*.
- Computing *C[i,j]* requires that we evaluate *(j-i)* terms and select their minimum.
- The computation of each term takes time t_c, and the computation of *C[i,j]* takes time $(j-i)t_c$. Each entry in diagonal *l* can be computed in time lt_c.

Optimal Matrix-Parenthesization Problem

- The algorithm computes $(n-1)$ chains of length two. This takes time $(n-1)t_c$; computing $n-2$ chains of length three takes time $(n-2)t_c$. In the final step, the algorithm computes one chain of length n in time $(n-1)t_c$.
- It follows that the serial time is $\Theta(n^3)$.

Optimal Matrix-Parenthesization Problem



The diagonal order of computation for the optimal matrix-parenthesization problem.

Parallel Optimal Matrix-Parenthesization Problem

- Consider a logical ring of processors. In step *l*, each processor computes a single element belonging to the l^{th} diagonal.
- On computing the assigned value of the element in table *C*, each processor sends its value to all other processors using an all-to-all broadcast.
- The next value can then be computed locally.
- The total time required to compute the entries along diagonal *l* is $lt_c+t_s log\ n+t_w(n-1)$.
- The corresponding parallel time is given by:

$$T_P = \sum_{l=1}^{n-1} (lt_c + t_s \log n + t_w(n-1)),$$

$$= \frac{(n-1)(n)}{2} t_c + t_s(n-1)\log n + t_w(n-1)^2.$$

Parallel Optimal Matrix-Parenthesization Problem

- When using *p* (<*n*) processors, each processor stores *n/p* nodes.
- The time taken for all-to-all broadcast of *n/p* words is

$$t_s \log p + t_w n(p-1)/p \approx t_s \log p + t_w n$$

and the time to compute *n/p* entries of the table in the l^{th} diagonal is $lt_c n/p$.

$$T_P = \sum_{l=1}^{n-1} (lt_c n/p + t_s \log p + t_w n),$$

$$= \frac{n^2(n-1)}{2p} t_c + t_s(n-1)\log p + t_w n(n-1).$$

$$T_P = \Theta(n^3/p) + \Theta(n^2)$$

- This formulation can be improved to use up to *n(n+1)/2* processors using pipelining.

Discussion of Parallel Dynamic Programming Algorithms

- By representing computation as a graph, we identify three sources of parallelism: parallelism within nodes, parallelism across nodes at a level, and pipelining nodes across multiple levels. The first two are available in serial formulations and the third one in non-serial formulations.

- Data locality is critical for performance. Different DP formulations, by the very nature of the problem instance, have different degrees of locality.